Preprint typeset in JHEP style. - HYPER VERSION

hep-th/0004113

The Discrete Bound State Spectrum of the Rotating D0-brane System

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ABSTRACT: In this note we obtain the discrete spectrum of the rotating ellipsoidal membrane, the solution to classical equations of motion in the matrix mechanics of N D0-branes. This solution has the interpretation of a closed D2-brane with the D0-branes bound to its surface. The semi-classical quantization is performed on the rotational modes with the result that both radii and energy are quantized. We also argue that the quantum mechanics of this system is well defined, with a unique ground state of positive energy in each sector with a non-zero angular momentum. The scaling of the size and energy of these states allows us to identify our rotating brane excited states with the previously conjectured resonances in the scattering of D0-branes.

KEYWORDS: D0-brane bound states, M-theory, Supermembrane.

1. Introduction

In this note we elaborate on our previous suggestion in [1] , and obtain the discrete spectrum of the rotating ellipsoidal membrane, the solution to classical equations of motion in the matrix mechanics of N D0-branes. This solution has the interpretation of a closed D2-brane with the D0-branes bound to its surface. We call it closed because it has the shape of an ellipsoid, which is in addition rotating transversely. Presumably such solution exists even in the decompactification limit, where it corresponds to a state of the fundamental eleven-dimensional M2 brane. In the latter case the D0-brane charge in 10D appears just as a result of Kaluza-Klein reduction. Such a configuration would play a significant role in rest-frame M-theory of Witten [2].

The main purpose of this note is to quantize the rotational modes of the system, and we find the exact expression for the energy in terms of the angular momentum quantum numbers. We imposed the condition of semi-classical or Bohr-Sommerfeld quantization on the action-angle variables. We obtain the spectrum, *i.e.* the discrete energy levels of the system, by expressing the energy in terms of positive-integer quantum numbers. In radial coordinates, the quantization proceeds in two steps: first, angular action-angle variable is quantized by introducing the angular momentum quantum number l; and second, the radial motion is considered, producing the radial quantum number n_r. An essential difference with the well-understood examples of the hydrogen atom and the spatial harmonic oscillator arises: in both cases the result for the quantized energy levels turns out to depend only on the principal quantum number n, which is the sum of the two $n = n_r + l$. Moreover this procedure gives the exact answer. In our system, the first step is almost guaranteed to give the exact result because angular momentum operator is sure to appear in the full QM. Thus we expect our result to be most reliable in the case when $n_r = 0$, and we call this the "leading Regge" trajectory because it is the lowest mass state for a given angular momentum. The correction to the energy due to radial motion, as a function of the radial quantum number n_r is also sketched out in Section 4.

The D2-brane, being a three-dimensional object, may have at most three independent components of the angular momentum tensor M_{ij}. Choosing the coordinate system such that this tensor is in a canonical block-diagonal form, the non-zero components of M_{ij} are $M_{12} = -M_{21}$, $M_{34} = -M_{43}$ and $M_{56} = -M_{65}$. We show in Section 4 that the system can be reduced, by using angular momentum conservation, to the following three dimensional radial potential in cylindrical coordinates

$$V(\vec{r}) = \frac{\hat{l}_1^2}{r_1^2} + \frac{\hat{l}_2^2}{r_2^2} + \frac{\hat{l}_3^2}{r_3^2} + \alpha^2 \left(r_1^2 r_2^2 + r_2^2 r_3^2 + r_1^2 r_3^2 \right) \tag{1.1}$$

The extensively studied YM mechanics [3, 4] is usually considered with $l_1 = l_2 = l_3 = 0$, and a non-compact moduli space where $V(\vec{r}) = 0$, that leads [5, 6] to the spreading

of the “particle” wave-function along the axis and into the “channel”[1]. In the general case, with angular momentum turned on, the centrifugal potential repels the “particle” from the “channel”. This must lead to a stable ground state, localized at the absolute minimum of the potential V : the point where $\partial V/\partial\vec{r}=0$,

$$\begin{aligned}\hat{l}_1^2 &= \alpha^2 r_1^4(r_2^2+r_3^2)\\ \hat{l}_2^2 &= \alpha^2 r_2^4(r_1^2+r_3^2)\\ \hat{l}_3^2 &= \alpha^2 r_3^4(r_1^2+r_2^2)\end{aligned} \tag{1.2}$$

which is exactly the relation that we found in [1] for our classical solution. By solving the above equations for r_i, and then evaluating the energy we get in Section 5 the following result for the total energy of the system as a function of ${l_i}^2$, in a certain approximation,

$$\frac{l_s^3\,N^3}{g_s}E^3 = {l_1}^2\,{l_2}^2+{l_2}^2\,{l_3}^2+{l_1}^2\,{l_3}^2+\frac{5}{12}\,({l_1}^2+{l_2}^2+{l_3}^2)^2 \tag{1.3}$$

This result is subject to the restriction to states with vanishing radial quantum number $n_r^i=0$, *i.e.* for the leading Regge trajectory.

The exact result was also obtained, and we refer the reader to the Section 5 for the exact formula, and the conditions for using the approximating formula.

It is our expectation that the whole construction of the solution, and the quantized energy levels can be carried over to the M2-brane in eleven dimensions. One possible way to do this is to retell the story word-by-word in the BFSS [7] infinite-momentum frame. Our hope is however to apply these ideas to a rest frame fundamental M2-brane. The connection between M-theory with Planck mass $M_p^{-1}=g_s^{1/3}\,l_s$ on a circle of radius $R_{11}=g_s\,l_s$ and type IIA theory with g_s and $\alpha'=1/(2\pi l_s^2)$ tells us that had M-theory been one with a continuous spectrum, the only massive states in IIA would be the KK and wrapping modes on the circle, corresponding to the D0-brane and the fundamental string respectively. This may or may not be the case, but it is not unnatural to suggest that uncompactified M-theory should have an infinite tower of discrete massive states of its fundamental object, the M-brane.

2. The System

The effective action of N D0-branes for weak and slowly varying fields is the non-abelian SU(N) Yang-Mills action plus the Chern-Simons action (for the bosonic part). For weak fields the action is gotten by dimensionally reducing the action of 9+1 dimensional $U(N)$ Super Yang-Mills theory to 0+1 dimensions [8]. Up to a constant

[1]The widely held belief that the YM quantum mechanics, the D0-brane system, the supermembrane and consequently M-theory all have a continuous spectrum is largely based on this observation.

term it is

$$S = -T_0(2\pi l_s^2)^2 \int dt \ \mathrm{Tr}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) \quad , \tag{2.1}$$

where $F_{\mu\nu}$ is the non-abelian $U(N)$ field strength in the adjoint representation and $T_0 = (g_s l_s)^{-1}$ is the D0-brane mass. To write this action in terms of coordinate matrices X^i, one has to use the dictionary

$$A_i = \frac{1}{2\pi l_s^2}X^i, \quad F_{0i} = \frac{1}{2\pi l_s^2}\dot{X}^i, \quad F_{ij} = \frac{-i}{(2\pi l_s^2)^2}[X^i, X^j] \tag{2.2}$$

with $i, j = 1, 2, ..., 9$, giving

$$S = T_0 \int dt \ \mathrm{Tr}\left(\frac{1}{2}\dot{X}^i\dot{X}^i + \frac{1}{4}\frac{1}{(2\pi l_s^2)^2}[X^i, X^j][X^i, X^j]\right) \tag{2.3}$$

To derive this it is necessary to gauge the A_0 potential away, which is possible for a non-compact time. The Gauss constraint

$$[\dot{X}^i, X^i] = 0 \tag{2.4}$$

persists, and the above action should be taken together with it [9].

The rest of this section is devoted to the discussion of the relation between currents constructed out of YM fields, and the corresponding membrane quantities.

Recently it has become clear [10, 11] that even though the D0-brane world-volume is only one-dimensional, a multiple D0-brane system can also couple to the brane charges of higher dimension. The Chern-Simons action derived in [12, 10, 11] for the coupling of N D0-branes to bulk RR $C^{(1)}$ and $C^{(3)}$ fields is

$$S_{CS} = T_0 \int dt \, \mathrm{Tr}\left(C_0 + C_i\dot{X}^i + \frac{1}{2\pi l_s^2}\left[C_{0ij}[X^i, X^j] + C_{ijk}[X^i, X^j]\dot{X}^k\right]\right) \tag{2.5}$$

This Chern-Simons action not only tells us how N D0-branes move in the weak background fields of Type IIA supergravity, but also what higher-form fields the D0-branes produce. Using this, one can see that it is possible to build p-branes of Type IIA string theory out of D0-branes [13, 10, 11].

The idea that a lower-dimensional object under the influence of higher-form RR fields may nucleate into spherically or cylindrically wrapped D-brane was proposed by Emparan [14] for the case of fundamental string. Before that, Callan and Maldacena [15] constructed a D- or F-string as a BI soliton solution on the D-brane where the attached string appears as a spike on the brane.

In general the D2-brane couples to the bulk RR field through the well-known CS coupling

$$S_{CS} = \int C^3 = \int C_{\mu\nu\rho}J^{\mu\nu\rho}d^3\sigma \tag{2.6}$$

where J is a three form RR current

$$J^{\mu\nu\rho} = \epsilon^{\alpha\beta\gamma}\ \partial_\alpha X^\mu \partial_\beta X^\nu \partial_\gamma X^\rho \quad , \tag{2.7}$$

or in form notation

$$^*J^{\mu\nu\rho}_{(3)} = dX^\mu \wedge dX^\nu \wedge dX^\rho \quad . \tag{2.8}$$

One can introduce a charge corresponding to this current, such that it is a world-volume 2-form even though it has three space-time indices, same as the current

$$\begin{aligned} Q^{\mu\nu\rho}_{\beta\gamma} &= X^{[\mu}\partial_\beta X^\nu \partial_\gamma X^{\rho]} \quad , \quad \text{so that} \\ J^{\mu\nu\rho}_{\alpha\beta\gamma} &= 3\,\partial_{[\alpha} Q^{\mu\nu\rho}_{\beta\gamma]} \quad , \end{aligned} \tag{2.9}$$

or in form notation

$$\begin{aligned} Q^{\mu\nu\rho}_{(2)} &= \left(dX^{[\mu} \wedge dX^\nu\right) X^{\rho]} \quad , \quad \text{such that} \\ dQ^{\mu\nu\rho}_{(2)} &= J^{\mu\nu\rho}_{(3)} \quad , \end{aligned} \tag{2.10}$$

where the exterior derivative is taken with respect to world-volume indices. This charge first appeared in the theory of bosonic relativistic membrane [16]. There it was interpreted as a topological charge of the membrane. The connection between RR charges and D-branes was discovered by Polchinski [17].

Instead of (2.7) one can represent the current as a Poisson bracket with respect to the spatial world-volume coordinates. For a static membrane, the non-zero components of $J^{\mu\nu\rho}$ are

$$J^{0ij} = \left\{X^i, X^j\right\} \quad , \tag{2.11}$$

For a moving membrane the completely spatial components also appear. A convenient generalization is, in the static gauge

$$J^{ijk} = \dot{X}^i \left\{X^j, X^k\right\} \quad . \tag{2.12}$$

The above discussion is for the ordinary D2-brane, and the coupling of the D0-brane matrix-mechanical system to the $C^{(3)}$ field is completely analogous

$$S_{CS} = \frac{T_0}{2\pi l_s^2} \int dt\ \ \mathrm{Tr}\left(C_{0ij}[X^i, X^j] + C_{ijk}[X^i, X^j]\dot{X}^k\right) = \int C \cdot J\ dt \tag{2.13}$$

The trace of the first term is zero identically, reflecting the fact that the total bare RR charge of the object that we are considering is zero, due to cancellation of the pieces with opposite orientation on the 2-sphere. It was proposed in [12, 11], that one needs to expand the expression in powers of X, to effectively obtain the multipole expansion of $C^{(3)}$. Then, it is possible to integrate by parts, to get instead the coupling of the field strength to what we shall call D2-brane dipole moment Q:

$$S_{CS} = \frac{T_0}{2\pi l_s^2} \int dt\ F_{0ijk}\, \mathrm{Tr}\left[X^i, X^j\right] X^k = \int F \cdot Q\ dt \tag{2.14}$$

3. The Rotating Membrane Solution

Next, we review a new kind of rotating solution to the system of N D0-branes, which was constructed in our previous work [1]. There we also showed that it is a stable object, in the sence of stability under small perturbations of initial conditions. Moreover it has interesting physical properties due to its dynamical nature, like the radiation of various SUGRA fields. The basic idea in the construction is that the attractive force of tension should be cancelled by the centrifugal repulsion force. The motion is at all times transverse, and in no way can be gauged away by coordinate reparametrization invariance on the membrane. We refer the reader to the original paper for more detail.

The D2-brane, being a three-dimensional object, may have at most 3 independent components of the angular momentum tensor M_{ij}. Choosing the coordinate system such that this tensor is in a canonical block-diagonal from, the non-zero components of M_{ij} are $M_{12} = -M_{21}$, $M_{34} = -M_{43}$ and $M_{56} = -M_{65}$. This argument is the same as for a particle in a central field, which always moves in a plane, and so has only one non-zero component of angular momentum.

Therefore, the freedom to choose the coordinate system, combined with the global SU(N) rotations, means that the very special ansatz we considered, may in fact describe the dynamics under much more general initial conditions. Hopefully, it will capture all the essential features of stationary states of the closed membrane.

We now review the construction of the rotating ellipsoidal membrane, viewed as a non-commutative collection of moving D0-branes, described in the non-relativistic limit by YM classical mechanics. For that we need to take the basic configuration [18] of the non-commutative fuzzy sphere in the 135 directions, and set it to rotate in the transverse space along three different axis, *i.e.* in the 12, 34 and 56 planes. We thus use a total of 6 space dimensions to embed our D-brane system. The corresponding ansatz is

$$\begin{aligned} X_1(t) &= \frac{2}{\sqrt{N^2-1}}\,\mathbf{T}_1\, r_1(t) \quad , \qquad X_2(t) = \frac{2}{\sqrt{N^2-1}}\,\mathbf{T}_1\, r_2(t) \quad , \\ X_3(t) &= \frac{2}{\sqrt{N^2-1}}\,\mathbf{T}_2\, r_3(t) \quad , \qquad X_4(t) = \frac{2}{\sqrt{N^2-1}}\,\mathbf{T}_2\, r_4(t) \quad , \\ X_5(t) &= \frac{2}{\sqrt{N^2-1}}\,\mathbf{T}_3\, r_5(t) \quad , \qquad X_6(t) = \frac{2}{\sqrt{N^2-1}}\,\mathbf{T}_3\, r_6(t) \quad . \end{aligned} \tag{3.1}$$

where the $N \times N$ matrices $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3$ are the generators of the N dimensional irreducible representation of $SU(2)$, with the algebra

$$[\mathbf{T}_i, \mathbf{T}_j] = i\,\epsilon_{ijk}\mathbf{T}_k \quad . \tag{3.2}$$

In this new ansatz we are using the matrix structure such that the coordinate matrices are proportional to the $SU(2)$ generators in pairs. Simultaneously, the Gauss constraint (2.4) is identically satisfied.

We interpret this as a rotation because one could make a rotation in *e.g.* the 12 plane which makes one of the components vanish, say X_2, while the other one gets a radius $r_1' = \sqrt{r_1^2 + r_2^2}$. This is possible exactly because both are proportional to the same matrix $\mathbf{T}_1$. The end result is that at any point in time one can choose a coordinate system in which the object spans only three space dimensions.

Substituting the ansatz into (2.3) gives the Hamiltonian

$$H = \frac{NT_0}{3}\left(\frac{1}{2}\sum_{i=1}^{6}\dot{r}_i^2 + \frac{\alpha^2}{2}\left[\left(r_1^2+r_2^2\right)\left(r_3^2+r_4^2\right) + \left(r_1^2+r_2^2\right)\left(r_5^2+r_6^2\right) + \left(r_3^2+r_4^2\right)\left(r_5^2+r_6^2\right)\right]\right) \ , \tag{3.3}$$

where we have introduced the convenient parameter $\alpha = \frac{2}{\sqrt{N^2-1}}\frac{1}{2\pi l_s^2}$.

The corresponding equations of motion are

$$\begin{aligned}
\ddot{r}_1 &= -\alpha^2\left(r_3^2+r_4^2+r_5^2+r_6^2\right)r_1 \ , \quad & \ddot{r}_2 &= -\alpha^2\left(r_3^2+r_4^2+r_5^2+r_6^2\right)r_2 \ , \\
\ddot{r}_3 &= -\alpha^2\left(r_1^2+r_2^2+r_5^2+r_6^2\right)r_3 \ , \quad & \ddot{r}_4 &= -\alpha^2\left(r_1^2+r_2^2+r_5^2+r_6^2\right)r_4 \ , \\
\ddot{r}_5 &= -\alpha^2\left(r_1^2+r_2^2+r_3^2+r_4^2\right)r_5 \ , \quad & \ddot{r}_6 &= -\alpha^2\left(r_1^2+r_2^2+r_3^2+r_4^2\right)r_6 \ .
\end{aligned} \tag{3.4}$$

We have found the special solution to these equations, describing a rotating ellipsoidal membrane with three distinct principal radii R_1, R_2 and R_3

$$\begin{aligned}
r_1(t) &= R_1\cos(\omega_1 t+\phi_1) \ , \quad & r_2(t) &= R_1\sin(\omega_1 t+\phi_1) \ , \\
r_3(t) &= R_2\cos(\omega_2 t+\phi_2) \ , \quad & r_4(t) &= R_2\sin(\omega_2 t+\phi_2) \ , \\
r_5(t) &= R_3\cos(\omega_3 t+\phi_3) \ , \quad & r_6(t) &= R_3\sin(\omega_3 t+\phi_3) \ .
\end{aligned} \tag{3.5}$$

This particular functional form of the solution ensures that the highly non-linear equations for any of the components r_i are reduced to a harmonic oscillator. The solution (3.5) keeps $r_1^2+r_2^2 = R_1^2$, $r_3^2+r_4^2 = R_2^2$ and $r_5^2+r_6^2 = R_3^2$ fixed which allows us to say that the object described by (3.5) rotates in six spatial dimensions as a whole without changing its basic shape.

Using the equations of motion (3.4), the three angular velocities are determined by the radii, and do not necessarily have to coincide:

$$\omega_1 = \alpha\sqrt{R_2^2+R_3^2} \ , \quad \omega_2 = \alpha\sqrt{R_1^2+R_3^2} \ , \quad \omega_3 = \alpha\sqrt{R_1^2+R_2^2} \ . \tag{3.6}$$

This dependence of the angular frequency on the radii is such that the repulsive force of rotation has to be balanced with the attractive force of tension in order for (3.5) to be a solution. Thus the radii R_1, R_2 and R_3 parameterize (3.5) along with the three phases ϕ_i, to produce altogether a six parameter family of solutions.

Next, we evaluate the energy,

$$E = \frac{NT_0}{4}\left(\omega_1^2R_1^2 + \omega_2^2R_2^2 + \omega_3^2R_3^2\right) \ . \tag{3.7}$$

In order to exhibit the properties of the solution (3.5) we compute the components of the angular momentum

$$M_{ij} = \mathrm{Tr}\left[X^i\Pi^j - X^j\Pi^i\right] \quad , \quad \text{where} \quad \Pi^i = T_0\dot{X}^i \quad . \tag{3.8}$$

As expected, the non-zero components are time-independent and equal to

$$M_{12} = \frac{1}{3}NT_0\omega_1R_1^2 \quad , \quad M_{34} = \frac{1}{3}NT_0\omega_2R_2^2 \quad , \quad M_{56} = \frac{1}{3}NT_0\omega_3R_3^2 \quad . \tag{3.9}$$

The angular momenta M_{12}, M_{34} and M_{56} correspond to rotations in the 12, 34 and 56 planes respectively. Their values (3.9) fit with the interpretation of the solution as N D0-branes rotating as an ellipsoidal membrane in that they are time-independent due to conservation law and proportional to $NT_0\omega_iR_i^2$.

4. Quantum Mechanics

The Hamiltonian (3.3) can be readily used to consider the corresponding quantum mechanical problem. However, a more thorough investigation of the procedure of reduction to radial coordinates (see below) is necessary in order to completely describe the D0-branes in 9+1 dimensions.

The picture is that of a particle in a six-dimensional space, and with a positive definite potential energy. It does not have an isolated minimum of the energy, but has instead a moduli space which is non-compact. The wave function spreads in those directions in the moduli space, the result being that there is no normalizable ground state with a positive energy. This does not mean that the theory is sick, but rather that it has a superimposed continuum spectrum, in addition to the discrete. In fact, the existence of this continuum is required if the system is to reduce to supergravity in the low-energy limit.

The conservation of angular momentum allows us to reduce the problem to a three-dimensional one in radial coordinates:

$$\begin{aligned} r_1 &= \rho_1\cos\phi_1 \ , \quad r_2 = \rho_1\sin\phi_1 \ , \\ r_3 &= \rho_2\cos\phi_2 \ , \quad r_4 = \rho_2\sin\phi_2 \ , \\ r_5 &= \rho_3\cos\phi_3 \ , \quad r_6 = \rho_3\sin\phi_3 \ . \end{aligned} \tag{4.1}$$

In these radial coordinates, the six-dimensional Laplacian can be broken up into radial and angular parts, with the angular part being computable from (3.9)

$$\left[-\left(\frac{\partial^2}{\partial\rho_1^2} + \frac{\partial^2}{\partial\rho_2^2} + \frac{\partial^2}{\partial\rho_3^2}\right) + \alpha^2(\rho_1^2\rho_2^2 + \rho_2^2\rho_3^2 + \rho_1^2\rho_3^2) + \frac{\hat{l}_1^2}{\rho_1^2} + \frac{\hat{l}_2^2}{\rho_2^2} + \frac{\hat{l}_3^2}{\rho_3^2}\right]\Psi = 2\,E\Psi \tag{4.2}$$

where we have identified the classical values of the angular momentum components M_{ij} with the corresponding quantum mechanical operator $\hat{l}$. In the non-degenerate

case, when at least two of the l's are non-zero, the above potential has an isolated global minimum at

$$\begin{aligned} \hat{l}_1^2 &= \alpha^2 \rho_1^4 (\rho_2^2 + \rho_3^2) \\ \hat{l}_2^2 &= \alpha^2 \rho_2^4 (\rho_1^2 + \rho_3^2) \\ \hat{l}_3^2 &= \alpha^2 \rho_3^4 (\rho_1^2 + \rho_2^2) \end{aligned} \tag{4.3}$$

The comparison with (3.9) and (3.6) shows that our classical solution (3.5) indeed is the one at the stable minimum of the potential. From here it is now clear that perturbations away from the equilibrium point are stable, and cause almost-harmonic oscillations around it. These can be analyzed by considering the tensor of the second order derivatives of the potential in (4.2)

$$\begin{pmatrix} 6\,\frac{l_1^2}{\rho_1^4} + \alpha^2\left(\rho_2^2 + \rho_3^2\right) & 2\alpha^2 \rho_1 \rho_2 & 2\alpha^2 \rho_1 \rho_3 \\ 2\alpha^2 \rho_1 \rho_2 & 6\,\frac{l_2^2}{\rho_2^4} + \alpha^2\left(\rho_1^2 + \rho_3^2\right) & 2\alpha^2 \rho_2 \rho_3 \\ 2\alpha^2 \rho_1 \rho_3 & 2\alpha^2 \rho_2 \rho_3 & 6\,\frac{l_1^2}{\rho_3^4} + \alpha^2\left(\rho_1^2 + \rho_2^2\right) \end{pmatrix} , \tag{4.4}$$

At the equilibrium point of (4.3) this is equal to

$$4\alpha^2 \begin{pmatrix} 2(\rho_2^2 + \rho_3^2) & \rho_1 \rho_2 & \rho_1 \rho_3 \\ \rho_1 \rho_2 & 2\left(\rho_1^2 + \rho_3^2\right) & \rho_2 \rho_3 \\ \rho_1 \rho_3 & \rho_2 \rho_3 & 2\left(\rho_1^2 + \rho_2^2\right) \end{pmatrix} , \tag{4.5}$$

By diagonalizing this matrix, we obtain the frequencies of the oscillations around the stable minimum, as well as the energies of the corresponding harmonic oscillator quantum mechanical states. The eigenvalues are everywhere positive:

$$\begin{aligned} \lambda_1 &= 2\,\left({\rho_1}^2 + {\rho_2}^2 + {\rho_3}^2\right) \\ \lambda_{2,3} &= {\rho_1}^2 + {\rho_2}^2 + {\rho_3}^2 \pm \frac{1}{\sqrt{2}} \sqrt{({\rho_1}^2 - {\rho_2}^2)^2 + ({\rho_2}^2 - {\rho_3}^2)^2 + ({\rho_3}^2 - {\rho_1}^2)^2} \end{aligned}$$

This allows us to approximately evaluate the energies of the states that are close, but not lying on the leading Regge trajectory. The simple formula below gives a correction to the energy (3.7) due to the radial oscillations, in terms of the three radial quantum numbers n_r ,

$$\delta E = \sqrt{\lambda_1}\, n_r^1 + \sqrt{\lambda_2}\, n_r^2 + \sqrt{\lambda_3}\, n_r^3 \quad . \tag{4.6}$$

5. Energy Quantization

We have previously shown [1] that the rotating ellipsoidal membrane solution is classically stable. However the system constantly loses energy due to the semi-classical radiation of various supergravity waves. As a side, we estimate the life-time of the

excited states: it is the ratio of the kinetic energy of the system to the rate of radiation per unit time,

$$\tau_{1/2} = \frac{E}{P} \sim \frac{N\,T_0\,\omega^2 R^2}{\kappa^2 N^2\,T_0^2\,\omega^{12} R^4} = \left(\kappa^2 N\,T_0\,\omega^{10} R^2\right)^{-1} \quad . \tag{5.1}$$

The substitution of the various quantities, including the characteristic size R of the system, which will turn out to be the eleven-dimensional Planck length $g_s^{1/3} l_s$ yields

$$\tau_{1/2} \sim \frac{1}{g_s^5\,l_s}\,N^9 \quad . \tag{5.2}$$

Thus the states are not stable in the absolute sence, but nevertheless are long-lived, compared to the string scale. We emphasize again, that the intrinsic dynamics of the system dictates a zero width.

What will happen to the rotating ellipsoidal membrane after all available kinetic energy has been radiated? The radiated quanta carry away angular momentum as well, eventually going into $l = 0$ state. As discussed above, the sector with vanishing angular momentum has a continuous spectrum, so the system will, most likely, decay into N free D0-branes.

However, for non-vanishing angular momentum, the Bohr-Sommerfeld quantization rule tells us that the angular momentum should be quantized in units of the Planck constant.

We use a scheme, due to Ehrenfest, in which an adiabatic invariant of the form $\int p_i dx_i$, which turns out to be equal to the angular momentum, is quantized in units of the Planck constant,[2]

$$\begin{aligned}
\frac{1}{2\pi} I_1 &= \frac{1}{2\pi}\,\mathrm{Tr}\int \Pi_1 dX_1 + \Pi_2 dX_2 = M_{12} = \frac{1}{3} N T_0 \omega_1 R_1^2 = l_1 \quad , \\
\frac{1}{2\pi} I_2 &= \frac{1}{2\pi}\,\mathrm{Tr}\int \Pi_3 dX_3 + \Pi_4 dX_4 = M_{34} = \frac{1}{3} N T_0 \omega_2 R_2^2 = l_2 \quad , \\
\frac{1}{2\pi} I_3 &= \frac{1}{2\pi}\,\mathrm{Tr}\int \Pi_5 dX_5 + \Pi_6 dX_6 = M_{56} = \frac{1}{3} N T_0 \omega_3 R_3^2 = l_3 \quad .
\end{aligned} \tag{5.3}$$

So far we are safe, as the quantization of the angular momentum as integer values of l_i is expected to be an exact feature of the complete quantum mechanics.

We should combine these equations with (3.6) and solve for R_i. Then reexpress the energy (3.7) in terms of l_1, l_2 and l_3 only:

$$\begin{aligned}
\frac{l_s^3\,N\,(N^2-1)}{g_s}\,E^3 = l_1^2\,l_2^2 + l_2^2\,l_3^2 + l_1^2\,l_3^2 + \\
\frac{1}{12}\left(\zeta + \frac{\psi^2}{\zeta} - \psi\right)\left(\zeta + \frac{\psi^2}{\zeta} + 3\,\psi\right)
\end{aligned} \tag{5.4}$$

[2]The Planck constant $\hbar$ has been set to unity throughout.

where

$$\begin{aligned}\zeta^3 &= 54\,\chi - \psi^3 + 6\,\sqrt{81\,\chi^2 - 3\,\chi\,\psi^3} \\ \psi &= l_1^2 + l_2^2 + l_3^2 \;\text{ and }\; \chi = l_1^2 l_2^2 l_3^2\end{aligned} \tag{5.5}$$

The uncertainty is, however, in the total angular momentum. We have used the naive value, l^2, but one might wonder whether the exact result can be gotten by a different identification, such as $\hat{l}_i^2 = (l+1)^2$. One argument in favor of this is the WKB approximation formulae, which dictates adding a unity to the RHS of (5.3). In any case, it is not clear whether the semi-classical approximation is miraculously capable of yielding the exact result, as is the case in the hydrogen atom.

Nevertheless, the principal message is that the bound-state spectrum of the D0-brane system is indeed discreet, and that there is a mass gap in every sector with non-zero relative angular momentum.

The result (5.4) is quite a bit complicated, but we shall see that it simplifies under certain assumptions. In particular, let n_i be large but approximately equal to each other (meaning their mutual ratios are close to one), then

$$\frac{l_s^3\,N\,(N^2-1)}{g_s} E^3 = l_1^2\,l_2^2 + l_2^2\,l_3^2 + l_1^2\,l_3^2 + \frac{5}{12}\,(l_1^2 + l_2^2 + l_3^2)^2 \tag{5.6}$$

The opposite limit, that of widely disparate values of l_i, can also be addressed analytically and gives the following result

$$\frac{l_s^3\,N\,(N^2-1)}{g_s} E^3 = l_1^2\,l_2^2 + l_2^2\,l_3^2 + l_1^2\,l_3^2 + 2\,l_1\,l_2\,l_3\sqrt{l_1^2 + l_2^2 + l_3^2}\;\; . \tag{5.7}$$

The characteristic scale of R, the analog of the Bohr radius, is the eleven dimensional Planck length $g_s^{1/3} l_s$. The mass scale, an analog of the Rydberg constant,[3] is $g_s^{1/3} l_s^{-1} N^{-1}$. Both are features that were previously expected [19], esp. in the D0-brane scattering problem [20, 21, 22]. We would like to suggest the identification between our rotating mebrane states and the resonances conjectured to exist in that context. However, the important consequence from perturbative string calculations and YM mechanics (see also Bachas [23]) is that the D0-brane system may probe important eleven dimensional physics, even in the non-relativistic limit. Certainly, this lends credibility to our suggestion that the fundamental M2-brane may form the same kind of closed ellipsoidal rotating configurations. We stress that progress is evident in this approach, even without the advantages of going over to the infinite-momentum frame M(atrix) theory [7].

[3] We point out that the apparent incompatibility of the mass scale with the eleven dimensional Planck mass goes away after the "de-boosting" procedure, along with the N dependence.

6. Acknowledgments

I would like to thank J. Ambjørn and C. Callan Jr. for extensive discussions and useful comments on the draft of the paper. Also many thanks to T. Harmark, D. Kabat, R. Myers, P. Pouliot, W. Taylor IV and Z. Yin.

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